



Stability Of First Order Partial Differential Equation

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Abstract: *In this paper, we prove the Hyers-Ulam-Rassias stability of first order partial differential equation:*

$$p(x, t)u_x(x, t) + p_x(x, t)u(x, t) + r(x, t) = g(x, t, u(x, t)).$$

Keywords: Hyers - Ulam - Rassias stability, Partial differential equations, Banach's contraction principle.

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1 INTRODUCTION

In 1940, S. M. Ulam [17] gave a well-known talk on stability for several functional equations. Ulam spoke about a problem concerning the stability of group homomorphism. D. H. Hyers [5] provided a partial answer to this Ulam's problem, in 1941. In 1998, Alsina and Ger [3] investigated the HU stability of the differential equation $y' = y$. Then there has been number of publications on stability of solutions of differential equations [6, 7] and partial differential equations [8,9]. This is now known as Hyers Ulam (HU) stability and its various extensions has been named with additional word. One such extension is Hyers Ulam Rassias (HUR) stability.

In [10] and [11], HUR stability for linear differential operators of n^{th} order with non-constant coefficients was established. HUR stability for special types of non-linear equations have been studied in [1, 2, 12, 13]. In 2011, Gordji et al. [4], proved the HUR stability of non-linear partial differential equations by using Banach's Contraction Principle. HUR stability of second order partial differential

equation have been studied in [14]. In [15], HUR stability of the third order partial differential equation have been established by employing Banach's contraction principle. HUR stability of second order partial differential equation have been studied in [16]. In this paper, by using the result of [4], we prove the HUR stability of first order partial differential equation:

$$p(x, t)u_x(x, t) + p_x(x, t)u(x, t) + r(x, t) = g(x, t, u(x, t)). \quad (1.1)$$

Here $p, r : J \times J \rightarrow \mathbb{R}^+$ be a differentiable function at least once w. r. t. both the arguments and $p(x, t) \neq 0, \forall x, t \in J, J = [a, b]$ be a closed interval and $g : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

Definition 1.1: A function $u : J \times J \rightarrow \mathbb{R}$ is called a solution of equation (1.1) if $u \in C^1(J \times J)$ and satisfies the equation (1.1).

2. PRELIMINARIES

Definition 2.1: The equation (1.1) is said to be HUR stable if the following holds:

Let $\varphi : J \times J \rightarrow (0, \infty)$ be a continuous function. Then there exists a continuous function $\Psi : J \times J \rightarrow (0, \infty)$, which depends on φ such that whenever $u : J \times J \rightarrow \mathbb{R}$ is a continuous function with

$$|p(x, t)u_x(x, t) + p_x(x, t)u(x, t) + r(x, t) - g(x, t, u(x, t))| \leq \varphi(x, t), \quad (2.1)$$

there exists a solution $u_0 : J \times J \rightarrow \mathbb{R}$ of (1.1) such that

$$|u(x, t) - u_0(x, t)| \leq \Psi(x, t), \quad \forall (x, t) \in J \times J.$$

We need the following.

Banach Contraction Principle:

Let (Y, d) be a complete metric space, then each contraction map $T : Y \rightarrow Y$ has a unique fixed point, that is, there exists $b \in Y$ such that $Tb = b$. Moreover,

$$d(b, w) \leq \frac{1}{(1-\alpha)} d(w, Tw), \quad \forall w \in Y \quad \text{and } 0 \leq \alpha < 1.$$

Following the results from Gordji et al. [4], we establish the following result.

3. MAIN RESULT

In this section we prove the HUR stability of first order partial differential equation (1.1).

Theorem 3.1: Let $c \in J$. Let p and g be as in (1.1) with additional conditions:

(i) $p(x, t) \geq 1, \forall x, t \in J$.

(ii) $\varphi: J \times J \rightarrow (0, \infty)$ be a continuous function and $M: J \times J \rightarrow [1, \infty)$ be an integrable function.

(iii) Assume that there exists $\alpha, 0 < \alpha < 1$ such that

$$\int_c^x M(\tau, t) \varphi(\tau, t) d\tau \leq \alpha \varphi(x, t). \quad (3.1)$$

and

$$K(x, t, u(x, t)) = \{p(x, t)\}^{-1} \left[p(c, t) u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u(\tau, t)) d\tau \right] \quad (3.2)$$

Suppose that the following holds:

C1: $|g(\tau, t, l(\tau, t)) - g(\tau, t, m(\tau, t))| \leq M(\tau, t) |l(\tau, t) - m(\tau, t)|, \forall \tau, t \in J$ and

$$l, m \in C(J \times J).$$

C2: $u: J \times J \rightarrow \mathbb{R}$ be a function satisfying the inequality (2.1).

Then there exists a unique solution $u_0: J \times J \rightarrow \mathbb{R}$ of the equation (1.1) of the form

$$u_0(x, t) = (p(x, t))^{-1} \left[p(c, t) u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u_0(\tau, t)) d\tau \right]$$

such that

$$|u(x, t) - u_0(x, t)| \leq \frac{\alpha}{(1-\alpha)} \varphi(x, t), \quad \forall x, t \in J.$$

Proof: Consider

$$|p(x, t) u_x(x, t) + p_x(x, t) u(x, t) + r(x, t) - g(x, t, u(x, t))| = |\{p(x, t) u(x, t)\}_x + r(x, t) - g(x, t, u(x, t))|$$

From the inequality (2.1), we get

$$|\{p(x, t) u(x, t)\}_x + r(x, t) - g(x, t, u(x, t))| \leq \varphi(x, t).$$

$$\Rightarrow -\varphi(x, t) \leq \{p(x, t) u(x, t)\}_x + r(x, t) - g(x, t, u(x, t)) \leq \varphi(x, t). \quad (3.3)$$

$$\Rightarrow \{p(x, t) u(x, t)\}_x + r(x, t) - g(x, t, u(x, t)) \leq \varphi(x, t).$$

Integrating from c to x we get,

$$p(x, t) u(x, t) - p(c, t) u(c, t) - \int_c^x r(\tau, t) d\tau - \int_c^x g(\tau, t, u(\tau, t)) d\tau \leq \int_c^x \varphi(\tau, t) d\tau.$$

$$\begin{aligned} \Rightarrow p(x, t) \left\{ u(x, t) - \{p(x, t)\}^{-1} \left[p(c, t) u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u(\tau, t)) d\tau \right] \right\} \\ \leq \int_c^x \varphi(\tau, t) d\tau. \end{aligned}$$

$$\Rightarrow \left\{ u(x, t) - \{p(x, t)\}^{-1} \left[p(c, t)u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u(\tau, t)) d\tau \right] \right\} \\ \leq \{p(x, t)\}^{-1} \int_c^x \varphi(\tau, t) d\tau.$$

$$\Rightarrow \left\{ u(x, t) - \{p(x, t)\}^{-1} \left[p(c, t)u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u(\tau, t)) d\tau \right] \right\} \\ \leq \int_c^x \varphi(\tau, t) d\tau,$$

$$(\because p(x, t) \geq$$

1).

$$\Rightarrow \{u(x, t) - K(x, t, u(x, t))\} \leq \int_c^x \varphi(\tau, t) d\tau,$$

where $K(x, t, u(x, t))$ is given by (3.2).

Since $M: J \times J \rightarrow [1, \infty)$ be an integrable function, we have

$$\{u(x, t) - K(x, t, u(x, t))\} \leq \int_c^x \varphi(\tau, t) d\tau \leq \int_c^x M(\tau, t) \varphi(\tau, t) d\tau.$$

Using inequality (3.1) we have,

$$\{u(x, t) - K(x, t, u(x, t))\} \leq \int_c^x M(\tau, t) \varphi(\tau, t) d\tau \leq \alpha \varphi(x, t).$$

Thus

$$\{u(x, t) - K(x, t, u(x, t))\} \leq \alpha \varphi(x, t). \quad (3.4)$$

In a similar way, from the left inequality of (3.3), we obtain

$$-\{u(x, t) - K(x, t, u(x, t))\} \leq \alpha \varphi(x, t). \quad (3.5)$$

From the inequalities (3.4) and (3.5) we get,

$$|u(x, t) - K(x, t, u(x, t))| \leq \alpha \varphi(x, t). \quad (3.6)$$

Let Y be the set of all continuously differentiable functions $\gamma : J \times J \rightarrow \mathbb{R}$. We define a metric d and an operator T on Y as follows: For $l, m \in Y$

$$d(l, m) = \sup_{x, t \in J} \left| \frac{l(x, t) - m(x, t)}{\varphi(x, t)} \right|$$

and the operator

$$\begin{aligned}
 (Tm)(x, t) &= \{p(x, t)\}^{-1} \left[p(c, t)u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, m(\tau, t)) d\tau \right] \\
 &= K(x, t, m(x, t)).
 \end{aligned} \tag{3.7}$$

Consider,

$$\begin{aligned}
 d(Tl, Tm) &= \sup_{x, t \in J} \left\{ \frac{Tl(x, t) - Tm(x, t)}{\varphi(x, t)} \right\}. \\
 &= \sup_{x, t \in J} \left\{ \frac{\{p(x, t)\}^{-1} \left(\int_c^x g(\tau, t, l(\tau, t)) d\tau - \int_c^x g(\tau, t, m(\tau, t)) d\tau \right)}{\varphi(x, t)} \right\}. \\
 &\leq \sup_{x, t \in J} \left\{ \frac{\left(\int_c^x g(\tau, t, l(\tau, t)) d\tau - \int_c^x g(\tau, t, m(\tau, t)) d\tau \right)}{\varphi(x, t)} \right\}, \quad (\because p(x, t) \geq 1). \\
 &\leq \sup_{x, t \in J} \left\{ \frac{\int_c^x |g(\tau, t, l(\tau, t)) - g(\tau, t, m(\tau, t))| d\tau}{\varphi(x, t)} \right\}.
 \end{aligned}$$

By using condition C1 we get,

$$\begin{aligned}
 d(Tl, Tm) &\leq \sup_{x, t \in J} \left\{ \frac{\int_c^x \{M(\tau, t) |l(\tau, t) - m(\tau, t)|\} d\tau}{\varphi(x, t)} \right\}. \\
 &= \sup_{x, t \in J} \left\{ \frac{\int_c^x \left\{ M(\tau, t) \varphi(\tau, t) \left(\frac{|l(\tau, t) - m(\tau, t)|}{\varphi(\tau, t)} \right) \right\} d\tau}{\varphi(x, t)} \right\}. \\
 &\leq \sup_{x, t \in J} \left\{ \frac{\int_c^x \left\{ M(\tau, t) \varphi(\tau, t) \times \sup_{\tau, t \in J} \left(\frac{|l(\tau, t) - m(\tau, t)|}{\varphi(\tau, t)} \right) \right\} d\tau}{\varphi(x, t)} \right\}.
 \end{aligned}$$

$$d(Tl, Tm) \leq d(l, m) \times \sup_{x, t \in J} \left\{ \frac{\int_c^x \{M(\tau, t) \varphi(\tau, t)\} d\tau}{\varphi(x, t)} \right\}.$$

By using inequality (3.1) we get,

$$d(Tl, Tm) \leq \alpha d(l, m).$$

By using Banach contraction principle, there exists a unique $u_0 \in X$ such that

$Tu_0 = u_0$, that is

$$\begin{aligned}
 \{p(x, t)\}^{-1} \left[p(c, t)u(c, t) + \int_c^x r(\tau, t) d\tau + \int_c^x g(\tau, t, u_0(\tau, t)) d\tau \right] &= u_0(x, t), \\
 &\text{(by using equation (3.7))}
 \end{aligned}$$

and

$$d(u_0, u) \leq \frac{1}{(1-\alpha)} d(u, Tu). \tag{3.8}$$

Now by using inequality (3.6) we get,

$$\begin{aligned}
 |u(x, t) - K(x, t, u(x, t))| &\leq \alpha \varphi(x, t). \\
 |u(x, t) - (Tu)(x, t)| &\leq \alpha \varphi(x, t). \\
 \Rightarrow \frac{|u(x, t) - (Tu)(x, t)|}{\varphi(x, t)} &\leq \alpha. \\
 \Rightarrow \sup_{x, t \in J} \frac{|u(x, t) - (Tu)(x, t)|}{\varphi(x, t)} &\leq \alpha.
 \end{aligned}$$

Thus

$$d(u, Tu) \leq \alpha. \quad (3.9)$$

Again

$$d(u_0, u) = \sup_{x,t \in J} \left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right|.$$

From equation (3.8) we get,

$$d(u_0, u) \leq \frac{1}{(1-\alpha)} d(u, Tu).$$

$$\sup_{x,t \in J} \left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right| \leq \frac{1}{(1-\alpha)} d(u, Tu).$$

$$\left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right| \leq \sup_{x,t \in J} \left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right| \leq \frac{1}{(1-\alpha)} d(u, Tu).$$

$$\left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right| \leq \frac{1}{(1-\alpha)} d(u, Tu).$$

From equation (3.9) we get,

$$\left| \frac{u_0(x,t) - u(x,t)}{\varphi(x,t)} \right| \leq \frac{1}{(1-\alpha)} \alpha.$$

$$|u_0(x, t) - u(x, t)| \leq \frac{\alpha}{(1-\alpha)} \varphi(x, t), \quad \forall x, t \in J.$$

Hence the result.

4. CONCLUSION

In this paper we have proved the HUR stability of the first order partial differential equation (1.1) by employing Banach's contraction principle.

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