A Survey On Domination Parameters In Graphs

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Abstract

Domination is one of the rapidly developing areas of graph theory, with applications in chessboard problems, network design, communication systems, facility location, and related optimization problems. This paper presents a systematic overview of the fundamental concepts and results in domination theory. It introduces the basic notions of dominating sets, minimal dominating sets, private neighbors, and domination number, followed by several important variants including independent domination, total domination, connected domination, neighborhood connected domination, edge domination, connected edge domination, 2-domination, domatic domination, connected domatic domination, inverse domination, and inverse edge domination. Important bounds, characterizations, and results for standard graphs such as paths, cycles, complete graphs, and complete bipartite graphs are discussed. The historical development of these concepts and the contributions of several researchers are also outlined. The study provides the necessary theoretical foundation for subsequent investigations in domination-related graph parameters and their applications.

Keywords: Dominating set , Domination Number, Induced subgraph

1.INTRODUCTION

One of the fastest growing areas with in graph theory is the study of domination. Motivated by numerous applications much work has been done as the graph on the topic of domination. Although mathematical study of domination related problems about 100 years prior. De Jaenisch(1862) attempted to determine the minimum number of queens required to cover an nn chess board. Rouse Ball (1892) reported three basic types of problems that chess players studied during this time. These include the following:

1.1.Covering : Determine the minimum number of chess pieces of given type that are necessary to cover (attack) every square of an nn chess board.

1.2. Independent Covering : Determine the smallest number of mutually non attacking chess pieces of a given type that are necessary to dominate every square of an nn board.
1.3. Independence : Determine the maximum number of chess pieces of a given type that can be placed on an nn chess board such that no two pieces attack each other. Note that if the chess piece being considered is the queen, this type of problem is commonly known as the N-queens problem.

The study of domination in graphs was further developed in the late 1950's and 1960's beginning with Claude Berge (1958). Berge wrote a book on graph theory, in which he introduced the coefficient of external stability, which is now known as the domination number of a graph. Oystein Ore (1962) introduced the terms dominating set and domination number in his book on graph theory. the problems described above were studied in more detail by brother Yaglom and Yaglom(1964). Their studies resulted in solutions to some of these problems for rooks, knights, kings bishops. A decade later, Cockayne and Hedetniemi(1977) published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic. An excellent treatment of fundamentals of domination is given in the book by Haynes et al. (1998) and a survey of several advanced topics in given in the book edited by Haynes et al.(1998a). In this section we present some of the basic definitions and theorems.

Definition1.3.1: A set $S \subseteq V$ is said to be a dominating set of G if every vertex in V-S is adjacent to some vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The domination number $\Gamma(G)$ is the maximum cardinality of dominating set of G.

A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G.

Definition1.3.2: Let S be a set of vertices in a graph G and let $u \in S$. We say that a vertex v is a private neighbor of u (with respect to S) if $N[v] \cap S = \{u\}$.

We define the private neighbor set of u with respect to S by $P_n[u\ S] = \{v : N[v] \cap S = \{u\}\}$. If $X \subseteq E$ and $e_1 \in X$ then the private neighbor of e_1 with respect to X is defined by $P_n[e_1, X] = \{e_2, N(e_2) \cap X = \{e\}\}$.

Theorem1.3.3: (Ore, 1962) If a graph G has no isolated vertices, then $\gamma(G) \le \frac{p}{2}$.

Theorem1.3.4: (Walikar, 1979) for any graph G,

$$\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \le \gamma(G) \le p - \Delta(G).$$

Theorem1.3.5: For any graph G, $\gamma(P_p) = \left\lceil \frac{p}{3} \right\rceil$

Theorem1.3.6: For any graph G, $\gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil$

Definition1.3.7: A subset S of vertices of a graph G is said to be an independent set if no two vertices of S are adjacent in G.

A dominating set S of a graph G is called an independent dominating set of G if S is an independent in G. The independent domination number i(G) of a graph G is the minimum cardinality of an independent dominating set. The upper independent domination number or independence number of a graph G is the maximum cardinality of an independent dominating set of G and is denoted by $\beta_0(G)$.

Cockayne et al. (1980) introduced the concept of total domination in graphs.

Definition1.3.8: A dominating set S of G is called a total dominating set of G if the induced subgraph $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Theorem1.3.9: $\gamma_t(P_p) = \begin{cases} \dfrac{p}{2} & \text{If } p \equiv 0 \pmod 4 \\ \left\lfloor \dfrac{p}{2} \right\rfloor + 1 & \text{other wise} \end{cases}$

Theorem1.3.10: $\gamma_t(C_p) = \begin{cases} \dfrac{p}{2} + 1 & \text{If } p \equiv 2 \pmod 4 \\ \left\lceil \dfrac{p}{2} \right\rceil & \text{other wise} \end{cases}$

Sampath Kumar and Walikar (1979) introduced the concept of connected domination in graphs.

Definition1.3.11: A dominating set S of G is called a connected dominating set if the induced subgraph ⟨S⟩ is connected. The minimum cardinality of a connected dominating set of G is called connected domination number of G and is denoted by $\gamma_c(G)$.

Theorem1.3.12: (Sampath Kumar and Walikar, 1979) $\gamma_c(P_p)=p-2$.

Theorem1.3.13: (Sampath Kumar and Walikar, 1979) $\gamma_c(C_p)=p-2$.

Definition1.3.14: If G is connected graph then a vertex cut a G is a subset R of V with the pupeuty that the subgraph of G induced by V–R is disconnected. If G is not a complete graph, then the vertex connectivity number k(G) as the minimum cardinality of a vertex cut. If G is complete graph K_p it is known that k(G)=p–1.

Theorem1.3.15: (Paulraj Joseph and Arumugam, 1992) Let G be non trivial connected graph then $\gamma_c(G)+k(G)\le p$.

Arumugam and Sivagnanam (2010) introduced the concept of neighborhood connected domination.

Definition1.3.16: A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph ⟨N(S)⟩ is connected. The minimum cardinality of ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$.

Theorem1.3.17: (Arumugam and Sivagnanam, 2010) For a path P_p, $\gamma_{nc}(P_p) = \left\lceil \dfrac{p}{2} \right\rceil$.

Theorem1.3.18: (Arumugam and Sivagnanam, 2010) For the cycle C_p of p vertices

$$\gamma_{nc}(C_p) = \begin{cases} \left\lceil \dfrac{p}{2} \right\rceil & \text{If } p \not\equiv 3 \pmod 4 \\ \left\lfloor \dfrac{p}{2} \right\rfloor & \text{If } p \equiv 3 \pmod 4 \end{cases}$$

Theorem1.3.19: (Arumugam and Sivagnanam, 2010) For the graph G, $\gamma_{nc}(G) \le \left\lceil \dfrac{p}{2} \right\rceil$.

As an analogy to vertex domination, the concept of edge domination was introduced by Mitchell and Hedetniemi (1977).

Definition1.3.20: A set S⊆E is said to be an edge dominating set of every edge in E-X is adjacent to some edge in X. The edge domination number of G is cardinality of a smallest edge dominating set of G and is denoted by $\gamma'(G)$.

Theorem1.3.21: For any path P_p, $\gamma'(P_p)=\left\lceil \frac{p-1}{3} \right\rceil$

Theorem1.3.22: For any cycle C_p, $\gamma'(C_p)=\left\lceil \frac{p}{3} \right\rceil$

Theorem1.3.23: For complete graph K_p, $\gamma'(K_p)=\left\lfloor \frac{p}{2} \right\rfloor$

Theorem1.3.24: For complete bipartite graph $K_{r,s}$, $\gamma'(K_{r,s})=\min\{r, s\}$.

Arumugam and Velammal (2009) introduced the concept of connected edge domination of a connected graph.

Definition1.3.25: An edge dominating set X of connected graph G is called a connected edge dominating set if the edge induced subgraph ⟨X⟩ is connected.

The minimum cardinality of a connected edge dominating set of G is called connected edge domination number and is denoted by $\gamma'_c(G)$.

Theorem1.3.26: (Arumugam and Velammal, 2009) For any path P_p, $\gamma'_c(P_p)=p-3$.

Theorem1.3.27: For a cycle C_p, $\gamma'_c(C_p)=p-2$.

The concept of 2-dominating was introduced by Fink and Jacobson in 1985.

Definition1.3.28: A subset D of V(G) is a 2-dominating set if every vertex not in S is adjacent to atleast 2 vertices of D. the 2-domination number $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G.

Theorem1.3.29: For a cycle C_p on $p\geq 3$ vertices $\gamma_2(C_p)=\left\lceil \frac{p}{2} \right\rceil$

The concept of Domatic number was introduced by Cockayne and Hedetniemi in 1977.

Definition1.3.30: Let G be a graph. A partition Δ of its vertex set V(G) is called a domatic partition of G if each class of Δ is a dominating partition of G if each class of Δ is a dominating set in G.

The minimum number of classes of a domatic partition of G is called the domatic number of G and is denoted by d(G) (E.J. Cockayne and S.T. Hedetniemi, in 1977)

Theorem1.3.31: For any graph G, $d(G)\leq\delta(G)+1$

A graph G is domatically full if $d(G)\leq\delta(G)+1$

If T is a tree with $p\geq 2$ vertices, then it is domatically full.

If C_p is a cycle of length p and p is divisible by 3, then C_p is domatically full.

The following result is based on a well-known result of Ore.

Theorem1.3.32: For any graph G, d(G)=1 if and only if G has an isolated vertex.

Theorem1.3.33: (E.J. (Cockyane and S.T. Hedetniemi in 1977) For any graph, $d(K_p+G)=p+d(G)$

Theorem1.3.34: (B. Zenlinka 1983) For any graph G, $\left\lfloor \frac{p}{(p-\delta(G))} \right\rfloor \le d(G)$.

The concept of connected domatic number of a graph was introduced by Laskar and Hedetniemi in 1984.

Definition1.3.35: A partition Δ of a vertex set V of a connected graph G is called a connected domatic partition of G if each class of Δ is a connected dominating set of G. The maximum number of connected domatic partition of G is called the connected domatic number of G and is denoted by $d_c(G)$.

Theorem1.3.36: For any connected graph G, $d_c(G) \le \delta(G)+1$

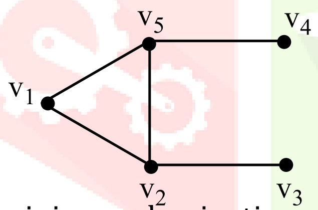
Theorem1.3.37: (Laskar and Hedetniemi in 1984) $d_c(G) \le \delta(G)+1$ if and only if $G=K_p$.

Theorem1.3.38: (Laskar and Hedetniemi in 1984) For any connected graph G which is not complete, $d_c(G) \le \delta(G)$.
Thus bound is sharp as can be seen with cycle C_4 The vertex connectivity number K(G) provides an upper bound on $d_c(G)$

Theorem1.3.39: (B. Zelinka in 1986) If G is a connected graph which not complete, then $d_c(G) \le K(G)$.
The concept of Inverse domination in graphs was introduced by Kulli and Sigarkanti in 1991.

Definition1.3.40: Let D be a minimum dominating set in graph G. If V-D contains a dominating set D of G, then D is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cordinality of inverse dominating set of G.

Ex: G:



The minimum dominating sets are {v_2, v_5}, {v_2, v_4} and {v_3, v_5} and the corresponding inverse dominating sets and {v_1, v_3, v_4}, {v_3, v_5} respectively thus $\gamma(G)=2$ and $\gamma^{-1}(G)=2$.

Note that every graph without isolated vertices contains an inverse dominating set.

Proposition1.3.41: If a graph G has no isolated vertices, then

$$\gamma(G) \le \gamma^{-1}(G)$$

Proposition1.3.42: If a graph G has no isolated vertices, then

$$\gamma(G)+\gamma^{-1}(G) \le p$$

Theorem1.3.43: (Kulli and Sigarkanti 1991)

Let D be a minimum dominating set of G. If for every vertex $v \in D$, the induced subgraph $\langle N[v] \rangle$ is a complete graph of order at least 2, then $\gamma(G)=\gamma^{-1}(G)$.

Theorem1.3.44: (Kulli and Sigarkanti 1991)

Let τ denote the family of minimum dominating set of G. If every minimum dominating set $D \in \tau$, V–D is independent, then $\gamma(G)+\gamma^{-1}(G) \le p$.

Theorem1.3.45: If a (p, q) graph G has no isolated vertices, then $\frac{(2p-q)}{3} \le \gamma^{-1}(G)$.

The concept of Inverse Edge Domination was introduced by Kulli and Soner in 1997.

Definition1.3.46: Let F be a minimum edge set of G. If E−F contains an edge dominating set F of G then F is called an inverse edge dominating set of G with respect to F. The inverse edge domination number $\gamma_e^{-1}(G)$ to be the minimum cardinality of an inverse edge domination set of G.

Clearly every inverse edge dominating set is an edge dominating set.

Proposition1.3.47: For any graph G without isolated vertices and edges . $\gamma'(G) \le \gamma_e^{-1}(G)$

Theorem1.3.48: (Kulli and Soner in 1997) . Let F be a minimum edge dominating set of G. If for each edge $e \in F$, the induced subgraph $\langle N(e) \rangle$ is a star then $\gamma'(G) = \gamma_e^{-1}(G)$.

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