

FUZZY PARTIAL DIFFERENTIAL EQUATION BY USING EXPLICIT METHOD

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Abstract

In this research, we consider an explicit method to solve fuzzy heat equation with boundary conditions. We express necessary materials and preliminaries, and we find a finite difference scheme for one dimensional heat equation with boundary conditions which include the integral equations approximated by using composite trapezoidal rule. Finally, an example illustrate the numerical results. In this example, we obtain the hausdroff distance between the exact solution and approximate solution .

Keywords: Fuzzy heat equation, finite difference, Fuzzy function.

I. INTRODUCTION

In this paper, we use the explicit method to solve the heat equation

$$(D_t - a^2 D_x^2) \tilde{U} = \tilde{0})$$

$$x \in (0,1), t \in (0,1]$$

nonlocal boundary conditions

with

$$\begin{cases} \tilde{U}(0, t) = \int_0^1 k_0(x) \tilde{U}(x, t) dx + \tilde{g}_0(t) \\ \tilde{U}(1, t) = \int_0^1 k_1(x) \tilde{U}(x, t) dx + \tilde{g}_1(t) \end{cases}$$

And the initial condition

$$\tilde{U}(x, 0) = \tilde{g}(x) \quad x \in (0,1)$$

Where, $\tilde{f}, \tilde{k}_0, \tilde{k}_1, \tilde{g}_0, \tilde{g}_1$ and $\tilde{g}$ are known fuzzy functions. The topics of numerical methods for solving fuzzy differential equations have been rapidly growing in recent years. The increase in research on fuzzy mathematics, necessitate the

study of fuzzy partial differential equations and fuzzy derivatives. The concept of fuzzy derivative was first derived by chang and zadeh in [4]. This discussion was followed by dubois and prade, who defined the extension principle in [5]. Some new method have been discussed by puri and Relescu in [6] and Goetschel and Voxman in [8]. In this work, we studied the fuzzy heat equation with nonlocal boundary conditions. we presented some preliminaries and finite difference method to derive the fuzzy heat equation, after that we use trapezoidal rule to estimate the integral terms. Finally, we presented an example.

II. PRELIMINARIES

We will use some notations to begin this section. Let X be a universal set, then the fuzzy set $\tilde{A}$ in X is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$. Where $\mu_{\tilde{A}}(x)$ is the membership grade of membership of x in $\tilde{A}$. The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite.

Definition 2.1 A fuzzy set $\tilde{A}$ in X and any real number α , then the α -cut set of $\tilde{A}$ is denoted by A_α and defined as

$$A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Similarly, $A'_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$ is called strong α -cut.

Definition 2.2 The triangular fuzzy number $\tilde{N}$ is defined by three numbers $\alpha < m < \beta$ as $\tilde{N} = (\alpha, m, \beta)$ and represented as bellow:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-\alpha}{m-\alpha} & \alpha \leq x \leq m \\ 1 & x = m \\ \frac{x-m}{\beta-m} & m < x \leq \beta \\ 0 & x = 0 \end{cases}$$

If $\alpha > 0$ ($\alpha \geq 0$), then $\tilde{A} > 0$ ($\tilde{A} \geq 0$).

If $\beta < 0$ ($\beta \leq 0$), then $\tilde{A} < 0$ ($\tilde{A} \leq 0$).

Definition 2.3 consider an arbitrary fuzzy number is shown, in the parametric form, by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$ with $0 \leq r \leq 1$ satisfying the following requirements:

1. $\underline{u}(r)$ is a bounded left semi continuous non-decreasing function over $[0,1]$
2. $\bar{u}(r)$ is a bounded left semi continuous non-increasing function over $[0,1]$
3. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

Peculiarly, if $\underline{u}$ and $\bar{u}$ are linear function, then we have a triangular fuzzy number. A crisp number u is represented by $\underline{u} = \bar{u} = u$, for all $0 \leq r \leq 1$.

Definition 2.4 For any fuzzy number $u = (\underline{u}(r), \bar{u}(r))$ and $v = (\underline{v}(r), \bar{v}(r))$, the algebraic operations are defined as bellow:

1. $ku = \begin{cases} (k\underline{u}, k\bar{u}) & k \geq 0 \\ (k\bar{u}, k\underline{u}) & k < 0 \end{cases}$
2. $u + v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$
3. $u - v = (\underline{u}(r) - \underline{v}(r), \bar{u}(r) - \bar{v}(r))$
4. $u \cdot v = (\min S, \max S)$, where $S = \{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}$.

Remark 2.1 since the α -cut of the fuzzy numbers is always a closed and bounded interval, then we can write $\tilde{A}_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$, for all $\alpha \in R$.

Definition 2.5 Assume that $u = (\underline{u}(r), \bar{u}(r))$ and $v = (\underline{v}(r), \bar{v}(r))$ are two fuzzy numbers. The Hausdorff metric D_H between u and v is defined by:

$$D_H(u, v) = \max_{r \in [0,1]} \{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}.$$

This metric is considered as a bound for computing error, and by using that the difference between exact solution and approximate solution is obtained.

III. FINITE DIFFERENCE METHOD

In this section, we use explicit method to solve the fuzzy heat equation

$$(D_t - \alpha^2 D_x^2) \tilde{U} = \tilde{0},$$

$$x \in (0,1), t \in (0,1] \quad \dots(3.1)$$

with the nonlocal boundary conditions

$$\begin{cases} \tilde{U}(0, t) = \int_0^1 k_0(x) \tilde{U}(x, t) dx + \tilde{g}_0(t) \\ \tilde{U}(1, t) = \int_0^1 k_1(x) \tilde{U}(x, t) dx + \tilde{g}_1(t) \end{cases}$$

$$\dots(3.2)$$

And the initial condition

$$\tilde{U}(x, 0) = \tilde{g}(x), \quad x \in (0,1) \quad \dots(3.3)$$

Where $\tilde{g}_0, \tilde{g}_1$ and $\tilde{g}$ are known fuzzy functions, $\tilde{U}$ is an unknown fuzzy function which must be determined and k_0 and k_1 are crisp known functions. Assume $\tilde{U}$ is a fuzzy function of the independent crisp variable x and t . define:

$$I = \{(x, t) | 0 \leq x \leq 1, 0 \leq t \leq T\}$$

α -cut of $\tilde{U}(x, t)$ and it's the parametric form, will be:

$$\tilde{U}(x, t)[\alpha] = [\underline{U}(x, t; \alpha), \bar{U}(x, t; \alpha)].$$

Assume that $\underline{U}(x, t; \alpha)$ and $\bar{U}(x, t; \alpha)$ have continuous partial differential, therefore $(D_t - \alpha^2 D_x^2) \bar{U}(x, t; \alpha)$ and $(D_t - \alpha^2 D_x^2) \underline{U}(x, t; \alpha)$ are continuous for all $(x, t) \in I$, $\alpha \in [0,1]$. Divide the domain $[0,1] \times [0,T]$ in to $M \times N$ mesh with spatial step size $h = \frac{1}{N}$ in x -direction and $k = \frac{T}{M}$ in t -direction. The gride points are given by:

$$\begin{aligned} x_i &= ih & i &= 0, 1, \dots, N \\ t_j &= jk & j &= 0, 1, \dots, M \end{aligned}$$

The value of $\tilde{U}$ at the representative mesh point $p(x_i, t_j)$ is denoted by

$$\tilde{U}_p = \tilde{U}(x_i, t_j) = \tilde{U}_{i,j}$$

The parameter form of fuzzy number $\tilde{U}_{i,j}$ is

$$\tilde{U}_{i,j} = (\underline{U}_{i,j}, \bar{U}_{i,j})$$

We have

$$\begin{cases} (D_t) \tilde{U}_{i,j} = (D_t \underline{U}_{i,j}, D_t \bar{U}_{i,j}) \\ (D_x^2) \tilde{U}_{i,j} = (D_x^2 \underline{U}_{i,j}, D_x^2 \bar{U}_{i,j}) \end{cases} \quad \dots(3.4)$$

Then by Taylor's expansion we obtain

$$\begin{cases} D_x^2 \underline{U}_{i,j} \approx \frac{\underline{U}_{i-1,j+1} - 2\underline{U}_{i,j+1} + \underline{U}_{i+1,j+1}}{h^2} \\ D_x^2 \bar{U}_{i,j} \approx \frac{\bar{U}_{i-1,j+1} - 2\bar{U}_{i,j+1} + \bar{U}_{i+1,j+1}}{h^2} \end{cases} \quad \dots(3.5)$$

And also, we have

$$\begin{cases} D_t \underline{U}_{i,j} \approx \frac{\underline{U}_{i,j+1} - \underline{U}_{i,j}}{k} \\ D_t \bar{U}_{i,j} \approx \frac{\bar{U}_{i,j+1} - \bar{U}_{i,j}}{k} \end{cases} \quad \dots(3.6)$$

The parametric form of heat equation will be

$$\begin{cases} D_t \underline{U}_{i,j} - \alpha^2 D_x^2 \bar{U}_{i,j} = \tilde{0} \\ D_t \bar{U}_{i,j} - \alpha^2 D_x^2 \underline{U}_{i,j} = \tilde{0} \end{cases} \quad \dots(3.7)$$

By (3.4) and (3.5) the difference scheme for heat equation is

$$\begin{cases} \frac{\underline{u}_{i,j+1} - \bar{u}_{i,j}}{k} - a^2 \frac{\bar{u}_{i-1,j+1} - 2\underline{u}_{i,j+1} + \bar{u}_{i+1,j+1}}{h^2} = 0 \\ \frac{\bar{u}_{i,j+1} - \underline{u}_{i,j}}{k} - a^2 \frac{\underline{u}_{i-1,j+1} - 2\bar{u}_{i,j+1} + \underline{u}_{i+1,j+1}}{h^2} = 0 \end{cases}$$

...(3.8)

By above equations we obtain

$$\begin{cases} -r\bar{u}_{i-1,j+1} + (1+2r)\underline{u}_{i,j+1} - r\bar{u}_{i+1,j+1} = \underline{u}_{i,j} \\ -r\underline{u}_{i-1,j+1} + (1+2r)\bar{u}_{i,j+1} - r\underline{u}_{i+1,j+1} = \bar{u}_{i,j} \end{cases}$$

...(3.9)

where $r = \frac{ka^2}{h^2}$

$\tilde{U} = (\underline{u}, \bar{u})$ is the exact solution of the approximating difference equations, $x_i, (i = 0, 1, \dots, N-1)$ and $t_j, (j = 0, 1, \dots, M)$.

We have $2(N-1)$ equations with $2(N+1)$ unknowns. Therefore we need four equations. We obtain these equations by nonlocal boundary conditions(3.2) are described by the trapezoid rule. So

$$\begin{aligned} a_0 \tilde{U}_{0,j+1} + \sum_{i=1}^{N-1} a_i \tilde{U}_{i,j+1} + a_N \tilde{U}_{i,j+1} &\approx -\tilde{g}_{0,j+1} \\ b_0 \tilde{U}_{0,j+1} + \sum_{i=1}^{N-1} b_i \tilde{U}_{i,j+1} + b_N \tilde{U}_{i,j+1} &\approx -\tilde{g}_{1,j+1} \end{aligned}$$

Where

$$a_0 = \frac{h}{2} k_0(x_0) - 1$$

$$a_N = \frac{h}{2} k_0(x_N)$$

$$b_N = \frac{h}{2} k_1(x_N) - 1$$

$$b_0 = \frac{h}{2} k_1(x_0)$$

and

$$a_i = hk_0(x_i), \quad b_i = hk_1(x_i) \quad i = 1, \dots, N-1$$

Also parametric form of fuzzy number $\tilde{g}_0$ and $\tilde{g}_1$ are:

$$\tilde{g}_0 = (\underline{g}_0, \bar{g}_0)$$

$$\tilde{g}_1 = (\underline{g}_1, \bar{g}_1)$$

obtain

$$\tilde{u}_{0,j+1} = Y^{-1}[Z_0(hk_1(x_N) - 2) - Z_1hk_0(x_N)]$$

$$\tilde{u}_{M,j+1} = Y^{-1}[Z_1(hk_0(x_0) - 2) - Z_0hk_1(x_0)]$$

Where,

$$Z_0 = -2h \sum_{i=1}^{N-1} k_0(x_i) \tilde{u}_{i,j+1} - 2\tilde{g}_{0,j+1}$$

$$Z_1 = -2h \sum_{i=1}^{N-1} k_1(x_i) \tilde{u}_{i,j+1} - 2\tilde{g}_{1,j+1}$$

and

$$Y = (hk_0(x_0) - 2)(hk_1(x_N) - 2) - h^2 k_1(x_0) k_0(x_N) \neq 0.$$

Equations(3.8) implies that:

$$\tilde{U}_{i,j+1} = r\tilde{U}_{i-1,j} + (1-2r)\tilde{U}_{i,j} + r\tilde{U}_{i+1,j},$$

$$i = 1, \dots, N-1, j = 0, 1, \dots, M$$

The approximate solution of fuzzy heat equation can be found.

IV. NUMERICAL RESULTS

Example:

In this section, we present a numerical example to illustrate our method, whose exact solution is known to us. Consider the fuzzy heat equation

$$\frac{\partial \tilde{U}}{\partial t}(x, t) - \frac{1}{\pi^2} \frac{\partial^2 \tilde{U}}{\partial x^2}(x, t) = \tilde{0}$$

$$0 < x < 1, t > 0$$

with the nonlocal boundary conditions

$$\tilde{U}(0, t) = \int_0^1 x \tilde{U}(x, t) dx + (1 + \frac{2}{\pi^2}) \exp(-t)$$

$$\tilde{U}(1, t) = \int_0^1 x \tilde{U}(x, t) dx - (1 - \frac{2}{\pi^2}) \exp(-t)$$

And the initial condition

$$\tilde{U}(x, 0) = \tilde{K} \cos \pi x$$

and $\tilde{K}[\alpha] = [\underline{k}(\alpha), \bar{k}(\alpha)] = [\alpha - 1, 1 - \alpha]$. which is easily seen to have exact solution for

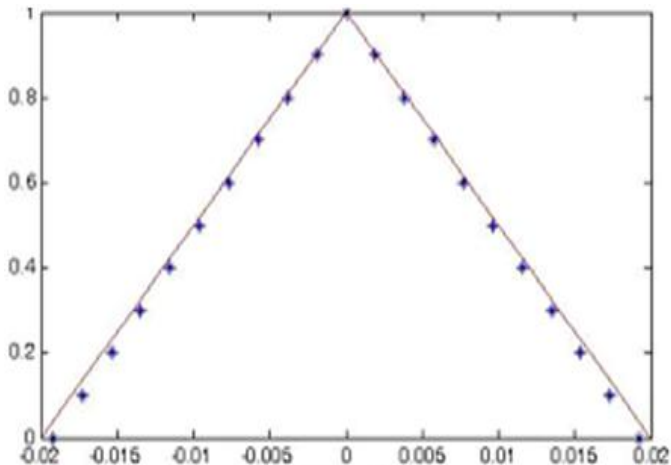
$$\frac{\partial \underline{U}}{\partial t}(x, t; \alpha) - \frac{1}{\pi^2} \frac{\partial^2 \underline{U}}{\partial x^2}(x, t; \alpha) = 0 - \alpha$$

$$\frac{\partial \bar{U}}{\partial t}(x, t; \alpha) - \frac{1}{\pi^2} \frac{\partial^2 \bar{U}}{\partial x^2}(x, t; \alpha) = 0 + \alpha$$

$$\text{are } \underline{U}(x, t; \alpha) = \begin{cases} \underline{k}(\alpha) \exp(-t) \cos \pi x & 0 < x < \frac{1}{2} \\ \bar{k}(\alpha) \exp(-t) \cos \pi x & \frac{1}{2} < x < 1 \end{cases}$$

$$\text{and } \bar{U}(x, t; \alpha) = \begin{cases} \bar{k}(\alpha) \exp(-t) \cos \pi x & 0 < x < \frac{1}{2} \\ \underline{k}(\alpha) \exp(-t) \cos \pi x & \frac{1}{2} < x < 1 \end{cases}$$

The exact and approximate solutions are shown in next figure at the point (0.2, 0.001) with $h=0.005$, $k=0.00001$. the hausdroff distance between solutions in this case is $7.58e - 004$.



$h=0.005, k=0.00001$

Conclusion

In this article, we presented an explicit method for fuzzy partial differential equation (FPDE) with integral terms. Finally, we given an numerical results. Also compared the hausdroff distance between the exact solution and approximate solution.

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