



Extreme Shock Maintenance Model Under Some Bivariate Replacement Policy By Partial Sum Process With Negligible Or Nonnegligible Repair Times

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Abstract: In this paper, Extreme Shock Maintenance Model Under Some Bivariate Replacement Policy by Partial Sum Process with Negligible or Nonnegligible Repair Times is studied. This study combines the NONN repair times model (Thangaraj and Rizwan, 2001) with extreme shock analysis to investigate maintenance optimization under bivariate replacement policy (T, N) Policy: Replace the system after a fixed working age (T) or upon N th failure. the problem is to determine an optimal replacement time $(T, N)^*$ such that the long-run average cost per unit time is minimized.

Keywords: Partial Sum Process, Replacement Policy, Shock models.

AMS Subject Classification: 60K10, 90B25.

I. INTRODUCTION

Reliability heavily relies on the study of maintenance problems. While most maintenance models focus solely on internal failure causes, they often overlook external factors. System failures can be triggered by external shocks. Shock models successfully applied in physics, communication, electronic engineering, and medicine, and also they are underutilized in analyzing deteriorating systems interrupted by random shocks. According to our model, system failure occurs when a single significant shock's damage surpasses a predetermined threshold. This type of shock is termed a 'deadly shock,' as its damage exceeds the threshold, leading to system failure. This concept is foundational to extreme shock models. Notably, Chen and Li (2008) explored maintenance strategies under the N -policy using an extreme shock model.

This study combines the NONN repair times model (Thangaraj and Rizwan, 2001) with extreme shock analysis to investigate maintenance optimization under bivariate replacement policy. This study derives the long-run average costs for multistate degenerative systems under bivariate replacement policy: (T, N) Policy: Replace the system after a fixed working age (T) or upon N th failure. This study establishes the existence of optimal solutions for bivariate replacement policy, ensuring system efficiency. The rest of the paper is organized as follows: In Section II, A comprehensive overview of the model is provided. In Section III, the long-run average cost per unit time is analytically derived for the model, exploring the impact of bivariate replacement policy. In Section IV, Conclusion is given.

II. The Model

Assumption 2.1: At initial time ($t = 0$), a new system is put into operation. Upon failure, the system undergoes repair. Eventually, it will be replaced by an identical new system some times later.

Assumption 2.2: The system operates amidst environmental shocks arriving according to a renewal process. The intervals between successive shocks, denoted as X_{ni} ($i = 1, 2, \dots$), are measured after the $(n-1)$ -st repair. The expected time between shocks is represented by $E(X_{11}) = \lambda$. Furthermore, the sequences X_{ni} are assumed to be independent and identically distributed (i.i.d.) for all $n \in \mathbb{N}$, ensuring consistency in the shock arrival pattern.

Assumption 2.3 : The shock damage sequence, Y_{ni} ($i = 1, 2, \dots$), represents the amount of damage sustained by the system from the i -th shock after the $(n-1)$ -st repair, with an expected value of $E(Y_{11}) = \mu$. Then $\{Y_{ni}, i = 1, 2, 3, \dots\}$ are iid sequences, for all $n \in \mathbb{N}$. If the system fails, it is closed, so that the random shocks have no effect on the system during the repair time. In the n -th operating stage, that is, after the $(n-1)$ -st repair, the system will fail, if the amount of the shock damage first exceed $(2^{n-1}\beta_0)M$, where $0 < \beta_0 \leq 1$ and M is a positive constant.

Assumption 2.4 : Let Z_n , $n = 1, 2, \dots$ be the repair time after the n -th repair and Z_n , $n = 1, 2, \dots$ constitute a non decreasing partial sum process with $E(Z_1) = \delta$ and ratio β_0 , such that $0 < \beta_0 < 1$. $N_n(t)$ is the counting process denoting the number of shocks after the $(n-1)$ -st repair. It is clear that $E(Z_n) = \delta \left(\frac{\mu}{2^{i-1}\beta_0} \right)$.

Assumption 2.5 : The system's economic parameters are defined by a reward rate of ' r ' per unit time when operating, a repair cost rate of ' c ' per unit time, and a replacement cost of R . Additionally, the replacement time, denoted as Z , is a random variable with an expected value of τ , $E(Z) = \tau$.

Assumption 2.6 : The sequences $\{X_{ni}, i = 1, 2, \dots\}$, $\{Y_{ni}, i = 1, 2, \dots\}$, $\{Z_n, n = 1, 2, \dots\}$ and Z are independent, ensuring statistical independence between shock patterns, damage, and replacement times.

Assumption 2.7: Define

$$\xi_n = \begin{cases} z_n & \text{if } z_n > 0 \\ 1 & \text{if } z_n = 0 \end{cases} \quad \text{for } n=1,2,3,\dots$$

Assumption 2.8 : The replacement policy (T, N) is adapted.

III. THE BIVARIATE REPLACEMENT POLICY (T, N)

In this section, we study an extreme shock model for the maintenance problem of a simple repairable system under (T, N) policy. Let

$$L_n = \min\{l; Y_{nl} > 2^{n-1}\beta_0 M\} \text{ and } W_n = \sum_{i=1}^{L_n} X_{ni}$$

Thus, L_n is the number of shocks until the first deadly shock occurred following the $(n-1)$ -st failure and L_n has a geometric distribution with $P\{L_n = k\} = p_n q_n^{k-1}$, $k = 1, 2, 3, \dots$

where $p_n = P(Y_{nl} > 2^{i-1}\beta_0 M)$ and $q_n = 1 - p_n$. We have $E(L_n) = \frac{1}{p_n}$. Since $\{X_{ni}, i = 1, 2, \dots\}$, $\{Y_{ni}, i = 1, 2, \dots\}$ are independent, it is clear that L_n are independent. Now

$$\begin{aligned} E(W_n) &= E\left(\sum_{i=1}^{L_n} X_{ni}\right) \\ &= E(L_n)E(X_{n1}) \\ &= \frac{\lambda}{p_n} \end{aligned}$$

The distribution function of W_n is $F_n(\cdot)$.

The Working age T of the system at time t is the cumulative life-time given by

$$T(t) = \begin{cases} t - V_n, & U_n + V_n \leq t \leq U_{n+1} + V_n \\ U_{n+1}, & U_{n+1} + V_n \leq t \leq U_{n+1} + V_{n+1} \end{cases}$$

where, $U_n = \sum_{k=1}^n W_k$ and $V_n = \sum_{k=1}^n Z_k$ and $U_0 = V_0 = 0$.

The replacement times are defined as follows: T_1 represents the first replacement time, and for $n = 2, 3, \dots$, T_n denotes the time between the $(n-1)$ -st replacement and the n -th replacement. Consequently, the sequence $\{T_n, n = 1, 2, \dots\}$ constitutes a renewal process. Cycles are defined by the time between consecutive system

replacements, spanning from initial installation to first replacement, and subsequent replacement intervals. Each replacement concludes one cycle and begins another. By combining the successive cycles with the costs associated with each, we establish a renewal reward process, facilitating the assessment of cumulative costs, rewards, and system efficiency.

According to the renewal reward theorem, the long-run average cost per unit time, $\mathcal{C}(T, N)$, for the replacement policy (T, N) is equal to the expected cost incurred in a renewal cycle divided by the expected cycle length, offering valuable insights into the policy's efficiency.

$$\begin{aligned}\mathcal{C}(T, N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\ &= \frac{\left[E\left\{ \left(c \sum_{n=1}^{\eta} \xi_n - rT \right) \chi_{(U_N > T)} \right\} + c_p E(Z) + \right]}{E(W)} \\ &\quad \left[E\left\{ \left(c \sum_{n=1}^{N-1} \xi_n - r \sum_{n=1}^N W_n \right) \chi_{(U_N \leq T)} \right\} + R \right] \quad (3.1)\end{aligned}$$

where $\eta = 0, 1, 2, \dots, N-1$ is the number of failures before the working age of the system exceeds T , W is the length of the cycle and $\chi(A)$ denotes the indicator function.

The length of the cycle under the replacement policy (T, N) is

$$W = (T + \sum_{n=1}^{\eta} \xi_n) \chi_{(U_N > T)} + (\sum_{n=1}^N W_n + \sum_{n=1}^{N-1} \xi_n) \chi_{(U_N \leq T)} + Z,$$

. Now

$$\begin{aligned}E \left[\left(\sum_{n=1}^N W_n \right) \chi_{(U_N \leq T)} \right] &= E \left[E \left(\sum_{n=1}^N W_n | U_N = u \right) \chi_{(U_N \leq T)} \right] \\ &= \int_0^T E \left(\sum_{n=1}^N W_n | U_N = u \right) dF_N(u) \\ &= \int_0^T E \left(\sum_{n=1}^N W_n \right) u dF_N(u) \\ &= \int_0^T \sum_{n=1}^N \frac{\lambda}{p_n} u dF_N(u) \\ &= \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u)\end{aligned}$$

The n-fold convolution of $F(\cdot)$ with itself is denoted by $F_N(\cdot)$, describing the distribution of the sum of n i.i.d. random variables, each governed by the distribution $F(\cdot)$.

$$\begin{aligned}E \left[\left(\sum_{n=1}^{N-1} \xi_n \right) \chi_{(U_N \leq T)} \right] &= E \left[E \left(\sum_{n=1}^{N-1} \xi_n | U_N = u \right) \chi_{(U_N \leq T)} \right] \\ &= \int_0^T E \left(\sum_{n=1}^{N-1} \xi_n | U_N = u \right) dF_N(u) \\ &= \int_0^T E \left(\sum_{n=1}^{N-1} \xi_n \right) dF_N(u)\end{aligned}$$

$$\begin{aligned}
&= \int_0^T \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) dF_N(u) \\
&= \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) \int_0^T dF_N(u) \\
&= \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T)
\end{aligned}$$

And hence

$$\begin{aligned}
E(W) &= E \left(\left(T + \sum_{n=1}^{\eta} \xi_n \right) \chi_{(U_N > T)} + \left(\sum_{n=1}^N W_n + \sum_{n=1}^{N-1} \xi_n \right) \chi_{(U_N \leq T)} + Z \right) \\
&= E[T \chi_{(U_N > T)}] + E \left[\left(\sum_{n=1}^{\eta} \xi_n \right) \chi_{(U_N > T)} \right] \\
&\quad + E \left[E \left(\sum_{n=1}^N W_n + \sum_{n=1}^{\eta} \xi_n \right) \chi_{(U_N \leq T)} \mid U_N = u \right] + E[Z] \\
&= T \overline{F_N(T)} + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) \\
&\quad + \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) + E \left[\left(\sum_{n=1}^{\eta} \xi_n \right) \chi_{((U_n < T < U_N))} \right] + \tau \\
&= T \overline{F_N(T)} + E \left[\left(\sum_{n=1}^{\eta} \xi_n \right) \chi_{((U_n < T < U_N))} \right] + \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) \\
&\quad + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) + \tau \\
&= T \overline{F_N(T)} + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) (F_n(T) - F_N(T)) \\
&\quad + \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) + \tau
\end{aligned}$$

The equation (3.1) becomes

$$\mathcal{C}(T, N) = \frac{\left[c \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) (F_n(T) - F_N(T)) \right.}{\left[T \overline{F_N(T)} + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) (F_n(T) - F_N(T)) \right.} \quad (3.2)$$

$$\left. - r \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) - r T \overline{F_N(T)} \right. \left. + c \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) \right. \left. + R + c_p \tau \right]$$

$$\left[+ \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) + \tau \right]$$

Theorem 3.1: For the model described in Section 2, under the assumptions 2.1 to 2.8, the long-run average cost per unit time under the bivariate replacement policy (T, N) for a simple degenerative repairable system is given by

$$\mathcal{C}(T, N) = \frac{\left[c \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) (F_n(T) - F_N(T)) \right.}{\left[T \overline{F_N(T)} + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) (F_n(T) - F_N(T)) \right.}$$

$$\left. - r \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) - r T \overline{F_N(T)} \right. \left. + c \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) \right. \left. + R + c_p \tau \right]$$

$$\left[+ \sum_{n=1}^N \frac{\lambda}{p_n} \int_0^T u dF_N(u) + \sum_{n=1}^{N-1} \left(\delta \left(\frac{\mu}{2^{n-1}\beta_0} \right) (1-p) + p \right) F_N(T) + \tau \right]$$

DEDUCTIONS

THE BIVARIATE REPLACEMENT POLICY (T, N) :

$\mathcal{C}(T, N)$ represents the long-run average cost as a bivariate function in T and N . Holding N constant, $\mathcal{C}(T, N)$ becomes a univariate function of T , denoted as $\mathcal{C}_m(T)$, when $N = m$. Obviously, when N is fixed, $\mathcal{C}(T, N)$ is a function of T . With N held constant at $N = m$, it can be written as

$$\mathcal{C}(T, N) = \mathcal{C}_m(T), \quad m = 1, 2, \dots$$

Thus, for a fixed m , we can find T_m^* by analytical or numerical methods such that $\mathcal{C}_m(T_m^*)$ is minimized. That is, when $N=1, 2, 3, \dots, m, \dots$, we can find $T_1^*, T_2^*, T_3^*, \dots, T_m^*, \dots$, respectively such that the corresponding $\mathcal{C}_1(T_1^*), \mathcal{C}_2(T_2^*), \mathcal{C}_3(T_3^*), \dots, \mathcal{C}_m(T_m^*), \dots$ are minimized. Because of the limited life span of multistate degenerative systems ensures a minimum average cost per unit time, so we can determine the minimum of the long-run average cost per unit time based on $\mathcal{C}_1(T_1^*), \mathcal{C}_2(T_2^*), \mathcal{C}_3(T_3^*), \dots, \mathcal{C}_m(T_m^*), \dots$

Then, if the minimum is denoted by $\mathcal{C}_n(T_n^*)$ we obtain the bivariate optimal replacement policy $(T, N)^*$ such that

$$\begin{aligned} \mathcal{C}(T, N)^* &= \min_N \mathcal{C}_n(T_n^*) \\ &= \min_N [\min_T \mathcal{C}(T, N)] \\ &\leq \min_N \mathcal{C}(\infty, N) = \mathcal{C}(N^*) \end{aligned}$$

The optimal policy $(T, N)^*$ is better than the optimal policy N^* , moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy T^* . So under the same conditions, an optimal policy $(T, N)^*$ is better than the optimal replacement policies N^* and T^* .

IV. Conclusion

This study investigates extreme shock maintenance models for multistate degenerative simple repairable systems, deriving explicit long-run average cost expressions for bivariate replacement policy: (T, N) . Optimality existence is established for these models.

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