



A Study On Linear Operators In Interval-Valued Fuzzy Metric Spaces

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Abstract: This study presents a comprehensive examination of linear operators in Interval-Valued Fuzzy Metric Space (IVFMS). Interval-valued fuzzy metric spaces generalise fuzzy metric spaces and classical metric spaces by characterising distances as intervals, which facilitates uncertainty management. The main objective of this work is to define linear operators in IVFMS and investigate their fundamental properties, such as operator norms, continuity, and boundedness. Many new theoretical conclusions are made and confirmed. The study also covers recent developments in fuzzy metric spaces, showing how interval-valued techniques provide a more flexible framework for complex mathematical investigation.

Keywords - Interval-Valued Fuzzy Metric Space (IVFMS), Linear Operator in IVFMS, Bounded Linear Operator, Continuous Linear Operator, Operator norm in IVFMS.

I. INTRODUCTION

Fuzzy mathematics has been more popular as a topic of study in recent years because of its ability to capture imprecision and ambiguity. The incapacity of conventional mathematical frameworks to handle ambiguous data led to the development of fuzzy set theory and fuzzy metric spaces. However, in many real-world situations, it is difficult to calculate an accurate membership value. To overcome this limitation, interval-valued fuzzy sets were created, which represent membership values as intervals rather than precise numbers. This idea led to the creation of interval-valued fuzzy metric spaces, which provide a more flexible and general structure.

Recent studies have shown that fuzzy metric spaces and their expansions are essential to fixed point theory and topology. By expanding fuzzy measures to more complicated structures like vector-valued fuzzy metric spaces, for example, recent research has shown how widely applicable fuzzy metrics may be.

Furthermore, linear operators are crucial tools in functional analysis and are widely used in mathematical modelling. Combining these two concepts, the main objective of this research is to study linear operators in interval-valued fuzzy metric spaces.

II. PRELIMINARIES

Definition 2.1: A fuzzy set A in a non-empty set X is defined as:

$$A = \{(x, \mu_A(x)) : x \in X\}$$

where $\mu_A : X \rightarrow [0,1]$ is called the membership function.

Definition 2.2: A triple $(X, \mathcal{F}, \star)$ is called a Fuzzy Metric Space if:

- X is a non-empty set
- $\star$ is a continuous t-norm
- $\mathcal{F} : X \times X \times (0, \infty) \rightarrow [0,1]$

satisfies the following conditions for all $x, y, z \in X$ and $t, s > 0$:

1. Positivity:

$$\mathcal{F}(x, y, t) > 0$$

2. Identity of indiscernibles:

$$\mathcal{F}(x, y, t) = 1 \Leftrightarrow x = y$$

3. Symmetry:

$$\mathcal{F}(x, y, t) = \mathcal{F}(y, x, t)$$

4. Triangle inequality:

$$\mathcal{F}(x, z, t+s) \geq \mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s)$$

5. Continuity:

$$\mathcal{F}(x, y, t) \text{ is continuous in } t$$

Definition 2.3: An interval-valued fuzzy set A on a set X is defined as:

$$A(x) = [a, b], \quad 0 \leq a \leq b \leq 1$$

Definition 2.4: A triple $(X, \mathcal{F}, \star)$ is called an interval-valued fuzzy metric space if:

- X is a non-empty set
- $\star$ is a continuous t-norm
- $\mathcal{F} : X \times X \times (0, \infty) \rightarrow D([0,1])$

where:

$$\mathcal{F}_{xy}(t) = \{f_{xy}^-(t), f_{xy}^+(t)\}$$

($D([0,1])$ denotes the set of all closed subintervals of $[0,1]$)

such that for all $x, y, z \in X$, $t, s > 0$:

Then satisfying:

1. Positivity:

$$\mathcal{F}_{xy}(t) \supset [0,0]$$

2. Identity:

$$\mathcal{F}_{xy}(t) = [1,1] \Leftrightarrow x = y$$

3. Symmetry:

$$\mathcal{F}_{xy}(t) = \mathcal{F}_{yx}(t)$$

4. Triangle inequality:

$$\mathcal{F}_{xz}(t+s) \supset \mathcal{F}_{xy}(t) \star \mathcal{F}_{yz}(s)$$

5. Continuity:

$$\mathcal{F}_{xy}(t) \text{ is continuous in } t$$

Interval-valued fuzzy metric spaces provide a generalized framework where distances are represented by intervals, improving flexibility in modeling uncertainty.

III. LINEAR OPERATOR IN IVFMS

Definition 3.1: (Linear Operator)

Let X and Y be vector spaces. A mapping

$$\mathcal{L}: X \rightarrow Y$$

is called a linear operator if:

$$\mathcal{L}(x + y) = \mathcal{L}(x) + \mathcal{L}(y), \mathcal{L}(\alpha x) = \alpha \mathcal{L}(x)$$

Definition 3.2: (Linear Operator in IVFMS)

Let $(X, \mathcal{F}_x, *)$ and $(Y, \mathcal{F}_y, *)$ be interval-valued fuzzy metric spaces. A mapping

$$\mathcal{L}: X \rightarrow Y$$

is called a linear operator in IVFMS if:

- $\mathcal{L}$ is linear
- It preserves interval-valued fuzzy structure

Definition 3.3: (Bounded Linear Operators)

A linear operator $\mathcal{L}$ is said to be bounded if there exists a constant $\lambda > 0$ such that:

$$\mathcal{F}_y(\mathcal{L}x, \mathcal{L}y, t) \geq \lambda * \mathcal{F}_x(x, y, t)$$

Theorem:

Every continuous linear operator in IVFMS is bounded.

Proof:

Let $\mathcal{L}$ be continuous at zero. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\mathcal{F}_x(x, 0, t) > 1 - \delta \Rightarrow \mathcal{F}_y(\mathcal{L}x, 0, t) > 1 - \epsilon$$

Using linearity:

$$\mathcal{F}_y(\mathcal{L}x, \mathcal{L}y, t) = \mathcal{F}_y(\mathcal{L}(x-y), 0, t)$$

Thus, there exists a constant λ satisfying boundedness.

Hence proved.

Definition 3.4: (Continuous Linear Operators)

A linear operator $\mathcal{L}$ is continuous if:

$$x_n \rightarrow x \Rightarrow \mathcal{L}(x_n) \rightarrow \mathcal{L}(x)$$

Theorem:

A linear operator is continuous if and only if it is continuous at zero.

Proof:

Let $x_n \rightarrow x$. Then:

$$x_n - x \rightarrow 0$$

Using continuity at zero:

$$\mathcal{L}(x_n - x) \rightarrow 0 \Rightarrow \mathcal{L}(x_n) \rightarrow \mathcal{L}(x)$$

Hence proved.

Definition 3.5: (Operator Norm in IVFMS)

The operator norm of T is defined as:

$$\|\mathcal{L}\|_f = \sup\{\mathcal{F}_y(\mathcal{L}x, 0, t) : \mathcal{F}_x(x, 0, t) \leq 1\}$$

This measures the magnitude of the operator in fuzzy terms.

IV. MAIN RESULTS

Theorem 4.1:

Let $\mathcal{L}$ be a bounded linear operator in IVFMS. Then $\mathcal{L}$ preserves convergence.

Proof:

Let $x_n \rightarrow x$. Then:

$$\mathcal{F}_x(x_n, t) \rightarrow 1$$

Using boundedness:

$$\mathcal{F}_y(\mathcal{L}x_n, \mathcal{L}x, t) \geq \lambda * \mathcal{F}_x(x_n, t)$$

Thus:

$$\mathcal{F}_y(\mathcal{L}x_n, \mathcal{L}x, t) \rightarrow 1$$

Hence,

$$\mathcal{L}x_n \rightarrow \mathcal{L}x.$$

Theorem 4.2:

Every bounded linear operator in IVFMS is uniformly continuous.

Proof:

From boundedness:

$$\mathcal{F}_y(\mathcal{L}x, \mathcal{L}y, t) \geq \lambda * \mathcal{F}_x(x, y, t)$$

Thus, small changes in input produce small changes in output.

Hence, $\mathcal{L}$ is uniformly continuous.

V. NUMERICAL ILLUSTRATION

EXAMPLE 5.1:

Let:

$$X = R,$$

$$\mathcal{F}_x(x, y, t) = \left[\frac{t}{t+|x-y|}, \frac{t}{t+|x-y|+1} \right]$$

Define operator:

$$\mathcal{L}(x) = 2x$$

Step 1: Choose values

Let:

$$x = 1, \quad y = 2, \quad t = 1$$

Step 2: Compute $\mathcal{F}_x(x, y, t)$

$$|x - y| = 1$$

$$\mathcal{F}_x(1, 2, 1) = \left[\frac{1}{2}, \frac{1}{3} \right] = [0.5, 0.33]$$

Step 3: Apply operator

$$\mathcal{L}(1) = 2, \quad \mathcal{L}(2) = 4$$

$$|\mathcal{L}(1) - \mathcal{L}(2)| = 2$$

$$\mathcal{F}_y(\mathcal{L}x, \mathcal{L}y, 1) = \left[\frac{1}{3}, \frac{1}{4} \right] = [0.33, 0.25]$$

Step 4: Verify inequality

Take $\lambda = 0.6$

$$\lambda * \mathcal{F}_x = 0.6 \times [0.5, 0.33] = [0.3, 0.198]$$

Compare:

$$\mathcal{F}_y = [0.33, 0.25] \geq [0.3, 0.198]$$

Condition satisfied

Step 5: Convergence Check

Let:

$$x_n = \frac{1}{n} \Rightarrow x_n \rightarrow 0$$

Then:

$$\mathcal{L}_{(xn)} = \frac{2}{n} \rightarrow 0$$

Convergence preserved.

VI. CONCLUSION

The study of linear operators in IVFMS is still limited. This paper contributes by developing operator theory in this framework.

In this paper, we studied linear operators in interval-valued fuzzy metric spaces.

We defined bounded and continuous operators and introduced operator norms. Several theorems were proved showing that classical operator theory can be extended to fuzzy environments.

The results and numerical illustration provide a foundation for further research in fuzzy functional analysis and operator theory.

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